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UNBALANCED BIPARTITE FACTORIZATIONS OF COMPLETE BIPARTITE GRAPHS

NIGEL MARTIN

ABSTRACT. We construct a new infinite family of factorizations of complete bipartite graphs by factors all of whose components are copies of a (fixed) complete bipartite graph $K_{p,q}$. There are simple necessary conditions for such factorizations to exist. The family constructed here demonstrates sufficiency in many new cases. In particular, the conditions are always sufficient when $q = p + 1$.

1. INTRODUCTION

We are dealing with simple graphs and use standard notation (see e.g. [13]). If G , H are two graphs, an H -factor of G is a spanning subgraph of G comprised of vertex-disjoint copies of H . An H -factorization of G is a decomposition of G into edge-disjoint H -factors. Here we study the case where $G = K_{m,n}$ and $H = K_{p,q}$ are complete bipartite graphs.

The existence of a factorization imposes a number of simple conditions on the values of m , n , p , q which provides straightforward arithmetical necessary conditions for such a factorization to exist.

Theorem 1. *If p , q are positive integers with $pq > 1$, then a necessary condition for the existence of a $K_{p,q}$ -factorization of $K_{m,n}$ is that the following quantities are all positive integers:*

$$\begin{aligned} \frac{m+n}{p+q}, \quad \frac{pm-qn}{p^2-q^2}, \quad \frac{pn-qm}{p^2-q^2}, \quad \frac{(pn-qm)n}{p(p-q)(m+n)}, \quad \frac{(pm-qn)n}{q(p-q)(m+n)} \\ \frac{(pm-qn)m}{p(p-q)(m+n)}, \quad \frac{(pn-qm)m}{q(p-q)(m+n)}, \quad \frac{mn(p+q)}{pq(m+n)} \quad \square \end{aligned}$$

Note that these are not all independent quantities. We call the conditions in Theorem 1 the Basic Arithmetic Conditions (BAC) of the problem for $K_{m,n}$ to be $K_{p,q}$ -factorizable. We have the following conjecture:

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BAC Conjecture *The necessary conditions stated in Theorem 1 are also sufficient.*

It turns out that for a fixed pair p, q and a fixed ratio $m : n$, there is a least pair (the base pair) m_0, n_0 in the same ratio, such that any m, n with this ratio satisfying the necessary conditions must be an integral multiple of the base pair. In addition, there is a simple construction that extends a factorization of the base pair to one of any multiple of it. Another simplification of the problem shows that we can assume that the pair p, q is coprime since $K_{sp,sq}$ -factorizations can be constructed from $K_{p,q}$ -factorizations in a convenient way. The details of these observations can be found in [4].

Considerable work has been done on this conjecture: The case $p = 1, q = 2$ was first proved in [9]. This was reproved by the author in [4] together with half the solution for $p = 1$ and $m = n$. The balanced case ($m = n$) has now been completed through a series of papers [5, 6, 7]. The general case for $p = 2, q = 3$ was solved in [12] and for $p = 1, q = 3$ in [8], which also covered a significant infinite family of situations.

From this point we assume that $\gcd(p, q) = 1$.

The most productive general method of construction seems to be to examine the way in which copies of $K_{p,q}$ are oriented in any factor of $K_{m,n}$. Let X, Y be the two sets of vertices making up the bipartition of $K_{m,n}$ ($|X| = m, |Y| = n$). In any $K_{p,q}$ -factor, there will be a number ξ of copies of $K_{p,q}$ with the q -set in X and the p -set in Y , and there will be a number η with this situation reversed. The ratio $\xi : \eta$ is common to each of the factors in the factorization. We call this the *balance ratio*. From this it is clear that $m = q\xi + p\eta, n = p\xi + q\eta$.

Further analysis of this (see [4]) shows that we can restate the necessary conditions for the base pair in another way via the balance ratio.

Theorem 2. *Let x_0, y_0 be the pair of integers such that $m_0 = qx_0 + py_0, n_0 = px_0 + qy_0$ is the base pair for the balance ratio $\xi : \eta$, then the pair x_0, y_0 is the least such in this ratio that the quantity $(p - q)x_0y_0/pq(x_0 + y_0)$ is an integer.* \square

An alternative formulation of this starts with a given *coprime* ratio pair and deduces the base pair.

Theorem 3. *Let $p < q$ and x, y be coprime pairs of positive integers ($p \neq q$), and let d be the denominator of the rational number $(q - p)xy/pq(x + y)$ expressed in lowest terms.*

Then the base pair for complete bipartite $K_{p,q}$ -factorizations with balance ratio $x : y$ is $m_0 = d(qx + py)$, $n_0 = d(px + qy)$. \square

Note that, in [4], the roles of x and y are reversed to the way presented here. This does not affect the thrust of the argument, however.

In [8] we proved the BAC Conjecture for an infinite family of cases where $p = 1$ and $\gcd(q - 1, x + y) = 2$. The aim of this paper is to prove it for all cases where $\gcd(q - p, x + y) = 1$. As a corollary, it follows that the BAC Conjecture is true wherever $q = p + 1$, a result previously also obtained by B. Du and J. Wang in [2]. The construction is given in Section 3.

To put this approach into context, however, we first go back to a rather impenetrable condition stated in [4] which described the extent to which the tiling construction in that paper was capable of solving the base case. A more detailed analysis in Section 2 results in a much simpler restatement of the condition, and shows that the result of the new construction in Section 3 is strictly stronger than before.

2. THE PLANAR TILING CONSTRUCTION

In [4], a planar tiling construction was used to give a very large infinite family of bipartite factorizations, and we were able to determine which of these gave a base case solution in the following way.

Given our $p < q$ and a balance ration $x : y$ (both coprime pairs), the construction of the planar tiling involves finding the unique coprime pair v, h such that $\frac{hp}{vq} = \frac{x}{y}$. Proposition 4.1 of [4] then establishes when the standard tiling gives the base case solution. In particular

Proposition 4. *Let $k = \gcd(vq + hp, \frac{(p-q)hv}{\gcd(vq, hp)})$. Then if $k = 1$ the BAC conjecture is true for $K_{p,q}$ -factorizations with balance ratio $x : y$. \square*

We examine the quantity k in more detail. First set $p_1 = \gcd(x, p)$, $p_2 = \gcd(y, p)$, $q_1 = \gcd(x, q)$, and $q_2 = \gcd(y, q)$ so that $x = p_1 q_1 x_0$, $y = p_2 q_2 y_0$, $p = p_1 p_2 p_0$, $q = q_1 q_2 q_0$. From the assumption that $\gcd(p, q) = \gcd(x, y) = 1$, it is straightforward to show that the quantities $p_1, p_2, q_1, q_2, x_0, y_0, p_0, q_0$ are all pairwise coprime apart from possibly (p_1, x_0) , (q_1, x_0) , (p_2, y_0) , (q_2, y_0) , (p_i, p_0) and (q_i, q_0) ($i = 1, 2$).

From the data

$$\frac{h}{v} = \frac{qx}{py} = \frac{q_1^2 x_0 q_0}{p_2^2 p_0 y_0}$$

and from coprimeness $h = q_1^2 x_0 q_0$ and $v = p_2^2 p_0 y_0$. Then $vq = p_2^2 q_1 q_2 p_0 y_0 q_0 = p_2 q_1 p_0 q_0 y$ and $hp = p_1 p_2 q_1^2 x_0 p_0 q_0 = p_2 q_1 p_0 q_0 x$ so,

$$\gcd(vq, hp) = p_2 q_1 p_0 q_0 \gcd(y, x) = p_2 q_1 p_0 q_0$$

Now we can recalculate the constant k in Proposition 4 as

$$\begin{aligned} k &= \gcd(vq + hp, \frac{(p-q)hv}{\gcd(vq, hp)}) \\ &= \gcd(p_2 q_1 p_0 q_0(x+y), \frac{(p-q)p_2^2 p_0 y_0 q_1^2 x_0 q_0}{p_2 q_1 p_0 q_0}) \\ &= p_2 q_1 \gcd(p_0 q_0(x+y), (p-q)x_0 y_0) \\ &= p_2 q_1 \gcd(x+y, (p-q)x_0 y_0) \\ &= p_2 q_1 \gcd(x+y, p-q) = \gcd(y, p) \gcd(x, q) \gcd(x+y, p-q) \end{aligned}$$

using the fact that $p_0|p$ and $\gcd(p_0, q) = \gcd(p_0, x_0) = \gcd(p_0, y_0) = 1$; similarly for q_0 . The last equality comes as $x_0|x$ and $\gcd(x, y) = 1$; similarly for y_0 . In summary we have

Theorem 5. *The planar tiling construction in [4] for $K_{p,q}$ -factorizations with balance ratio $x : y$ gives a factorization of $K_{km, kn}$ where $K_{m,n}$ is the base case and*

$$k = \gcd(y, p) \gcd(x, q) \gcd(x+y, p-q). \quad \square$$

Note that the roles of x and y may be reversed, if necessary, to improve the result.

3. THE CASE $\gcd(q-p, x+y) = 1$

Theorem 5 shows that $\gcd(q-p, x+y)$ is a key number and this is reinforced by the results of [8] where the value 2 is dealt with extensively [for the case $p = 1$].

Theorem 6. *The BAC Conjecture is true whenever $\gcd(q-p, x+y) = 1$.*

Proof. Without loss, we assume that $p < q$, with $\gcd(p, q) = \gcd(x, y) = 1$. As before, set $p = p_0 p_1 p_2$, $q = q_0 q_1 q_2$, $x = p_1 q_1 x_0$ and $y = p_2 q_2 y_0$, where $p_1 = \gcd(p, x)$, $p_2 = \gcd(p, y)$, $q_1 = \gcd(q, x)$ and $q_2 = \gcd(q, y)$.

From Theorem 3, we find that $d = p_0 q_0(x+y)$ so that the base case has $m = p_0 q_0(x+y)(qx+py) = p_0 q_0 p_1 q_2(x+y)\mu$, where $\mu = q_0 q_1^2 x_0 + p_0 p_2^2 y_0$, and $n = p_0 q_0(x+y)(px+qy) =$

$p_0q_0p_2q_1(x+y)\nu$, where $\nu = p_0p_1^2x_0 + q_0q_2^2y_0$. The factor size is $p_0q_0(x+y)^2pq$, which implies that there are $\mu\nu$ factors to account for.

The edges of $K_{m,n}$ correspond naturally with the places in an $m \times n$ matrix F that we call the *factor matrix*. We aim to populate the entries of F with values in the range $1, \dots, \mu\nu$ so that entries with the same value correspond to the edges of the respective $K_{p,q}$ -factors.

Each factor is a collection of copies of $K_{p,q}$. Those with q vertices in the m -set of the bipartition of $K_{m,n}$ are called *vertical* and those with q vertices in the n -set are called *horizontal*.

F will be built up in rectangular blocks of increasing sizes. We define first those relating to the vertical factor pieces.

Let J be a $q_0q_2\mu \times p_0p_2\nu$ matrix with general term $J_{\alpha\beta}$. We can express $\alpha = (a-1)q_0q_2 + c$ and $\beta = (b-1)p_0p_2 + d$ uniquely for a, b, c, d where $1 \leq a \leq \mu$, $1 \leq b \leq \nu$, $1 \leq c \leq q_0q_2$ and $1 \leq d \leq p_0p_2$. Set $J_{\alpha\beta} = (a-1)\nu + b$ and J is then decomposable as a $\mu \times \nu$ array of rectangular $q_0q_2 \times p_0p_2$ blocks (called *microblocks*) where each microblock has a single factor label and the labels read in the natural order across J from left to right and from top to bottom.

J is a model with rotational variants $J(i, j)$ called *miniblocks*. $J(i, j)$ is obtained by rotating the rows of J cyclically downwards iq_0q_2 places and the columns cyclically to the right jp_0p_2 places. The effect is to leave a microblock structure with labels shifted cyclically down i places and to the right j places. Note that $J = J(0, 0)$.

The following is a figurative example with $\mu = 5$, $\nu = 7$, each square is a microblock and has all its elements with the given label.

$$J(0, 0) = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ \hline 15 & 16 & 17 & 18 & 19 & 20 & 21 \\ \hline 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ \hline 29 & 30 & 31 & 32 & 33 & 24 & 35 \\ \hline \end{array}, J(1, 2) = \begin{array}{|c|c|c|c|c|c|c|} \hline 24 & 35 & 29 & 30 & 31 & 32 & 33 \\ \hline 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ \hline 13 & 14 & 8 & 9 & 10 & 11 & 12 \\ \hline 20 & 21 & 15 & 16 & 17 & 18 & 19 \\ \hline 27 & 28 & 22 & 23 & 24 & 25 & 26 \\ \hline \end{array}$$

Using these, we next construct the $p_0p_1q_0q_2\mu \times p_0p_2q_0q_1\nu$ matrix H as a $p_0p_1 \times q_0q_1$ block array of miniblock variants of J . Specifically, if $1 \leq \gamma \leq p_0p_1$ and $1 \leq \delta \leq q_0q_1$, then the miniblock of H of row index γ and column index δ is defined to be $J(\delta-1, \gamma-1)$.

Here is an example with $\mu = 5, \nu = 7, p_0p_1 = 3, q_0q_1 = 4$. The squares represent microblocks, the shaded one being the microblock with label 1, and the bolder subdividing lines indicating the miniblocks structure of H .

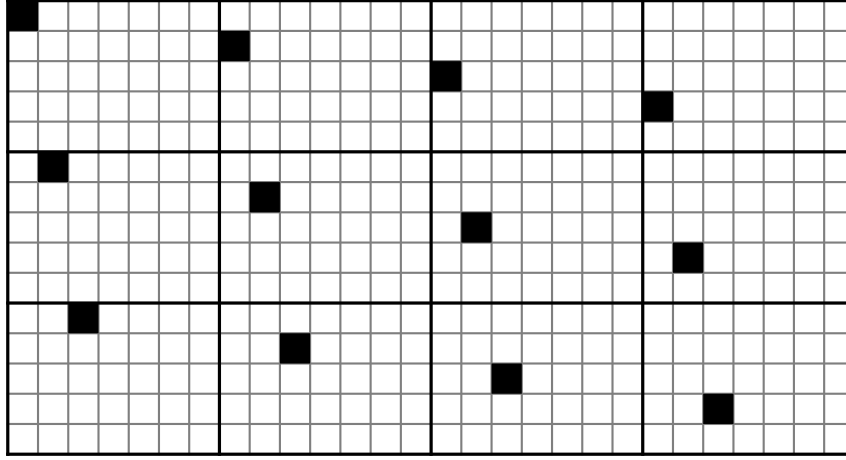


Figure 1

Note that $p_0p_1 < \nu$ and $q_0q_1 < \mu$, so that all the miniblocks comprising H are distinct (i.e. none can be a rotation back to $J(0,0)$). This ensures that H has the following properties:

- Lemma 7.** (1) *In every column and every row of microblocks in H , a given factor label is associated with at most one microblock.*
- (2) *Within every column (respectively row) of miniblocks in H , a given factor label is associated with a cyclically contiguous set of p_0p_1 (respectively q_0q_1) microblock columns (respectively rows).* \square

Here the term “cyclically contiguous” means a sequence of consecutive columns (or rows) where, if necessary, the leftmost (respectively top) microblock column (row) of a miniblock column (row) follows cyclically after the rightmost column (bottom row).

H also has rotational variants $H(i, j)$. For $0 \leq i < q_1x_0$ and $0 \leq j < p_1x_0$, $H(i, j)$ is the array of miniblocks where, for $1 \leq \alpha \leq p_0p_1$ and $1 \leq \beta \leq q_0q_1$, the miniblock with row index α and column index β is $H(i, j)_{\alpha\beta} = J(iq_0q_1 + \beta - 1, jp_0p_1 + \alpha - 1)$.

Lemma 8. *If $i \neq i'$, then in any fixed microblock row the set of factor labels occurring in $H(i, j)$ is disjoint from the set of factor labels occurring in $H(i', j)$. Similarly, if $j \neq j'$, in any fixed microblock column the set of factor labels occurring in $H(i, j)$ is disjoint from the set of factor labels occurring in $H(i, j')$.*

Proof. The restriction on the ranges of i, j , imply that $iq_0q_1 + \beta - 1 < \mu$ and $jp_0p_1 + \alpha - 1 < \nu$. Thus there is no chance of any “over rotation” to cause unwanted factor label duplication in rows or columns. \square

Now we can place the vertical factor pieces in F . First decompose F as an $(x+y) \times (x+y)$ array G of blocks of size $p_0p_1q_0q_2\mu \times p_0p_2q_0q_1\nu$ (i.e. of size equal to H). We first assign to the partial row G_{11}, \dots, G_{1x} . Recall that $x = p_1q_1x_0$. For $1 \leq j \leq x$ write $j - 1 = rp_1 + s$ uniquely for $0 \leq s < p_1$, $0 \leq r < q_1x_0$, and write $r = tq_1 + u$ uniquely for $0 \leq u < q_1$, $0 \leq t < x_0$. Then define $G_{1j} = H(r, s + tp_1)$. Finally, for $2 \leq i \leq (x+y)$, and $1 \leq v \leq x$, define $j \equiv i + v - 1 \pmod{x+y}$ in the range $1, \dots, x+y$, and $G_{ij} = G_{1v}$. This has the effect of copying the assignments of the top row of G by a process of diagonal replication into the other rows; i.e. as we go down one row we shift the assignment cyclically one place to the right.

The following detail of the sequence of H -type blocks in the first row of G may help to elucidate the structure of what is going on.:

$$\begin{aligned}
& H(0, 0), H(0, 1), \dots, H(0, p_1 - 1), \\
& H(1, 0), H(1, 1), \dots, H(1, p_1 - 1), \dots, \\
& H(q_1 - 1, 0), \dots, H(q_1 - 1, p_1 - 1), \\
& H(q_1, p_1), H(q_1, p_1 + 1), \dots, H(2q_1 - 1, 2p_1 - 1), \\
& H(2q_1, 2p_1), \dots, H(3q_1 - 1, 3p_1 - 1), \\
& \dots, H((x_0 - 1)q_1, (x_0 - 1)p_1), \dots, H((x_0 - 1)q_1, x_0p_1 - 1), \\
& \dots, H(x_0q_1 - 1, (x_0 - 1)p_1), \dots, H(x_0q_1 - 1, x_0p_1 - 1)
\end{aligned}$$

Lemma 9. *This definition determines an assignment of vertical copies of $K_{p,q}$ so that no two copies with the same factor label overlap in a column or in a row.*

Proof. Consider the factor label 1 as an exemplar. The construction of H shows 1 occupying the microblock columns of index $1 \dots p_0p_1$ in each miniblock column and rows of index $1 \dots q_0q_1$ in each miniblock row. In the rotational variant $H(i, j)$ these become microblock columns of index $jp_0p_1 + 1 \dots (j+1)p_0p_1$ and rows of index $iq_0q_1 + 1 \dots (i+1)q_0q_1$ respectively.

So in the actual construction in the partial first row of G , we have blocks $H(r, s + tp_1)$ where the microblock column coverage is $(s+tp_1)p_0p_1 + 1 \dots (s+tp_1+1)p_0p_1$ and microblock row coverage $r q_0q_1 + 1 \dots (r+1)q_0q_1$.

Then reading across left to right, repetitions in microblock rows can only occur when r is fixed, and there are precisely p_1 for each value of r . But a microblock occupies $p_0 p_2$ columns of F , so across any given microblock row where label 1 occurs, we have identified a microblock-row-subarray with $p_0 p_2 p_1 = p$ columns (and $q_0 q_2$ rows) in F .

Next, for fixed r , the microblock columns labelled 1 change to contiguous but non-overlapping microblock columns as the value of $s + t p_1$ changes. On the other hand, if r changes and $s + t p_1$ is fixed, the columns microblock coincide.

But, as can be seen from the extended listing above, reading left to right along a partial row of blocks in the G -structure these H -blocks come in groups of length p_1 . This means that the diagonal replication procedure for completing all the partial rows of G ensures that the microblock-row-subarrays stack on top of each other in such a way that their microblock columns are either disjoint or are identical, with the latter case occurring only when the value of $s + t p_1$ is the same. For any such value, this occurs (for the varying r , precisely q_1 times (the range of the variable u), so that combining them we get a subarray with total row coverage of size $q_0 q_2 q_1 = q$ in F to go with the column coverage of size p . So we have identified a collection of disjoint vertical copies of $K_{p,q}$ as required, for the label 1.

Finally, while the label 1 is an exemplar, it is typical because of the cyclical nature of the constructions. \square

Again we need to know the vertical and horizontal coverage of factor labels within miniblocks.

Lemma 10. (1) *Within every column (resp. row) of miniblocks in G thus far defined, a given factor label is associated with a cyclically contiguous set of $p_0 p_1^2 x_0$ (resp. $q_0 q_1^2 x_0$) microblock columns (resp. rows).*

(2) *$p_0 p_1^2 x_0 < \nu$ and $q_0 q_1^2 x_0 < \mu$ so there is no danger of the resulting contiguous sets of rows and columns rotating back onto themselves.*

Proof. From Lemma 7, we know that we have p_0p_1 microblock column coverage and q_0q_1 microblock row coverage in any G_{ij} . Looking along a defined row of G , we have $p_1q_1x_0$ such situations, with q_1x_0 groups where p_1 have identical coverage (being part of a $q \times p$ array of the relevant factor label). Each time a single rotation is made to enter the next one of these groups we cover another q_0q_1 rows that are cyclically contiguous to the previous set. Since this happens q_1x_0 times in all, we get the row coverage as stated.

Similarly for the column coverage. \square

This completes the definition of the vertical factor pieces. The approach for the horizontal pieces is similar, but we require a new miniblock structure.

1	35	2	29	3	30	4	31	5	32	6	33	7	34
34	1	35	2	29	3	30	4	31	5	32	6	33	7
7	34	1	35	2	29	3	30	4	31	5	32	6	33
8	7	9	1	10	2	11	3	12	4	13	5	14	6
6	8	7	9	1	10	2	11	3	12	4	13	5	14
14	6	8	7	9	1	10	2	11	3	12	4	13	5
15	14	16	8	17	9	18	10	19	11	20	12	21	13
13	15	14	16	8	17	9	18	10	19	11	20	12	21
21	13	15	14	16	8	17	9	18	10	19	11	20	12
22	21	23	15	24	16	25	17	26	18	27	19	28	20
20	22	21	23	15	24	16	25	17	26	18	27	19	28
28	20	22	21	23	15	24	16	25	17	26	18	27	19
29	28	30	22	31	23	32	24	33	25	34	26	35	27
27	29	28	30	22	31	23	32	24	33	25	34	26	35
35	27	29	28	30	22	31	23	32	24	33	25	34	26

Figure 2

Let a, b, c, d be positive mutually coprime integers. Construct an $ac \times bd$ array L as follows: For $1 \leq s \leq c, 1 \leq t \leq d$, assign the label $d(s-1) + t$ to the entries L_{ij} where $i = a(s-1) + \alpha, j = b(t-1) + \alpha$ for $1 \leq \alpha \leq ab$, where the subscripts are reduced modulo ac and bd respectively into the correct ranges.

Note that there are a total of $abcd$ assignments, which is the correct number, so we need to check that no entry of K is assigned more than once. This follows quickly from the coprime assumption.

The case with $a = 3$, $b = 2$, $c = 5$, $d = 7$ is shown in Figure 2; entries with label 1 have been emboldened.

In our situation we take $a = q_0q_2$, $b = p_0p_2$, $c = \mu$ and $d = \nu$ which we know to be mutually coprime. This is our standard model miniblock M . Note that the construction is such that any given label occupies a diagonal sequence of $q_0q_2p_0p_2$ entries in M and that looking rightwards by p_0p_2 generally increases the label by 1 while looking downwards by q_0q_2 generally increases the label by ν .

We construct cyclic variants $M(i, j)$ of M , by rotating columns $p_0p_1^2x_0p_0p_2 + iq_0q_2p_0p_2$ places cyclically to the right and rows $q_0q_1^2x_0q_0q_2 + jq_0q_2p_0p_2$ places cyclically down.

Lemma 10 defines columns and rows of miniblocks already covered by a given label. The effect of the summands $p_0p_1^2x_0p_0p_2$ and $q_0q_1^2x_0q_0q_2$ is to ensure that we avoid these in anything previously defined. The other two summand ensure that varying i and j avoids partial overlaps with the diagonals (of length $q_0q_2p_0p_2$)

From each miniblocks $M(i, j)$, we construct a larger block $L(i, j)$ as a $p_0p_1 \times q_0q_1$ array of miniblocks all equal to $M(i, j)$.

Lemma 11. (1) *Within every column (resp. row) of miniblocks in $L(i, j)$, a given factor label is associated with a cyclically contiguous set of $p_0p_2q_0q_2$ columns (resp. rows).*

(2) *For each factor label ϕ , the entries with that label contribute a total of $p_0p_2q_0q_2$ subarrays of size $p_0p_1 \times q_0q_1$, with non-overlapping rows and columns, but covering the contiguous sets as described above.* \square

We now use these to fill in the remaining G -blocks of F and to define all the required horizontal factors. The approach is to complete the first row of G and copy this over the remaining rows by diagonal replication.

So we define the $G_{1,x+j}$, $1 \leq j \leq y = p_2q_2y_0$. Working as in the vertical case, given j , we can define unique integers r , s , t where $j - 1 = rq_2 + s$, $r = tp_2 + u$, $0 \leq s < q_2$, $0 \leq u < p_2$, $0 \leq t < y_0$, from which we set $G_{1,x+j} = L(r, s + tq_2)$. We then extend this over the remainder of G by assigning $G_{\alpha\beta} = G_{1,\alpha-\beta+1}$ for $2 \leq \alpha \leq (x + y)$ and

$x + 1 \leq (\alpha - \beta + 1) \leq x + y$ where all the calculations are taken modulo $x + y$ in the range $1, \dots, x + y$.

Lemma 12. *This definition determines an assignment of horizontal copies of $K_{p,q}$ so that no two copies with the same factor label overlap in a column or in a row.*

Proof. This follows *mutatis mutandis* using the same argument as for the proof of Lemma 9. □

The corresponding analysis also proves the following.

Lemma 13. (1) *Within every column (resp. row) of miniblocks in G defined in the second stage, a given factor label is associated with a cyclically contiguous set of $q_0q_2^2y_0$ (resp. $p_0p_2^2y_0$) microblock columns (resp. rows).*
 (2) *$q_0q_2^2y_0 < \nu$ and $p_0p_2^2y_0 < \mu$ so there is no danger of the resulting contiguous sets of rows and columns rotating back onto themselves.*

Finally, we observe that the construction in the horizontal stage produces factor labellings that do not overlap vertically or horizontally with those created in the vertical stage since μ and ν are the respective sums of the two ranges of coverage calculated in Lemmas 10 and 13. It follows by counting together with the properties set out in Lemmas 7 to 13 that we must have constructed the factorization required. So the proof of Theorem 6 is complete. □

Corollary 14. *For all $p \geq 1$, the BAC-conjecture for $K_{p,p+1}$ factorizations is true.* □

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DURHAM, DURHAM DH1 3LE, ENGLAND

E-mail address: `nigel.martin@durham.ac.uk`